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THE TRANSPIRATION-COOLED FLAT PLATE WITH VARIOUS THERMAL AND VELOCITY BOUNDARY CONDITIONS

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THIS note is concerned with the effect of thermal and velocity boundary conditions on the heat-transfer characteristics of a transpiration-cooled flat plate. Consideration is given to a laminar, constant-property boundary-layer flow and a transpiring gas whose fluid properties are the same as those of the main stream gas. The various cases studied here are enumerated below.

(a) T_w = const., $v_w \sim x^{-\frac{1}{2}}$;

(b) T_w = const., v_w = const.;

(c) q_w = const., $v_w \sim x^{-\frac{1}{2}}$;

(d) q_w = const., v_w = const.;

(e) $q_w = \rho v_w c_p(T_w - T_c)$, v_w = const.

T_c is the temperature of the coolant gas in the supply reservoir behind the porous wall and the other symbols have their usual meaning.

Cases (a) through (d) are self-explanatory; however, Case (e) merits some amplification. The thermal boundary condition represents a balance between the convective heat transfer from the boundary layer into the wall and the enthalpy rise of the coolant gas as it passes through the porous wall. Such a balance is valid when extraneous heat losses are negligible. Thus, in Case (e), neither the surface temperature T_w nor the surface heat flux q_w is prescribed; rather, these quantities vary along the plate surface in a manner consistent with the dynamics of the boundary layer. It is believed that the boundary conditions that comprise Case (e) correspond most closely to those that can be achieved in experiment.

It is well established that Cases (a) and (c) yield similarity solutions of the boundary-layer momentum and energy equations. On the other hand, Cases (b), (d), and (e) do not possess similarity solutions. For these cases, the velocity and heat-transfer solutions are found by a series expansion method. For the velocity problem, one defines a stream function

$$\psi = \sqrt{(U_\infty \nu x)}\, f(\eta, \beta) - v_w x \qquad (1)$$

where

$$\eta = \frac{y}{2x}\sqrt{(Re_x)}, \quad \beta = \frac{2v_w}{U_\infty}\sqrt{(Re_x)}, \quad Re_x = \frac{U_\infty x}{\nu} \qquad (2)$$

η is the well-known similarity variable and, except for a difference in sign and a factor of 2, β is identical to the conventional blowing parameter f_w. The reduced stream function f is expressed as a series

$$f(\eta, \beta) = f_0(\eta) + \beta f_1(\eta) + \beta^2 f_2(\eta) + \beta^3 f_3(\eta) + \ldots \qquad (3)$$

For the thermal problem, it is convenient to define dimensionless temperature variables, respectively for Cases (b), (d), and (e), as follows

$$\vartheta = \frac{T_\infty - T}{T_\infty - T_w}, \quad \vartheta = \frac{T_\infty - T}{q_w/\rho c_p v_w}, \quad \vartheta = \frac{T_\infty - T}{T_\infty - T_c} \qquad (4)$$

The ϑ variable is also expanded in a series

$$\vartheta(\eta, \beta) = \vartheta_0(\eta) + \beta\vartheta_1(\eta) + \beta^2\vartheta_2(\eta) + \beta^3\vartheta_3(\eta) + \beta^4\vartheta_4(\eta) + \ldots \qquad (5)$$

The ordinary differential equations governing the functions $f_i(\eta)$ and $\vartheta_i(\eta)$ are derived by substituting the series into the momentum and energy equations. The corresponding boundary values are found by applying the boundary conditions for Cases (b), (d), and (e); in addition, $u = 0$ at $y = 0$ and $u \to U_\infty$ and $T \to T_\infty$ as $y \to \infty$ for all cases. Owing to space limitations, the detailed derivations will be omitted here.

The heat-transfer results can be expressed in terms of a local heat-transfer coefficient and a local Nusselt number defined as

$$h = q_w/(T_\infty - T_w), \quad Nu_x = hx/k \qquad (6)$$

in which q_w is positive when heat flows from the boundary layer into the wall. Nusselt numbers for Case (a) appear in the literature (see, for example, reference [1]), while the results for the other cases were obtained as part of this investigation.

Table 1 contains a listing of $Nu_x/\sqrt{(Re_x)}$ values for Cases (a) and (c); these follow directly from the similarity solutions. On the other hand, for Cases (b), (d), and (e), the Nusselt numbers are respectively expressed by the following relationships*

* The primes denote differentiation with respect to η.

Table 1. Nusselt number results for Cases (a) and (c); Pr = 0·7

$2(v_w/U_\infty)\sqrt{(Re_x)}$	$Nu_x/\sqrt{(Re_x)}$ Case (a)	Case (c)
0	0·2927	0·4059
0·1	0·2665	0·3772
0·2	0·2408	0·3485
0·3	0·2155	0·3196
0·5	0·1661	0·2612
0·7	0·1187	0·2014
1·0	0·05175	0·1052

$$Nu_x/\sqrt{(Re_x)} = -\frac{1}{2}[\vartheta_0'(0) + \beta\vartheta_1'(0) + \beta^2\vartheta_2'(0) + \beta^3\vartheta_3'(0)] \quad (7a)$$

$$= \frac{Pr}{2}[\vartheta_1(0) + \beta\vartheta_2(0) + \beta^2\vartheta_3(0) + \beta^3\vartheta_4(0)]^{-1} \quad (7b)$$

$$Nu_x/\sqrt{(Re_x)} = \frac{Pr}{2}\left[\frac{1 - \beta\vartheta_1(0) - \beta^2\vartheta_2(0) - \beta^3\vartheta_3(0)}{\vartheta_1(0) + \beta\vartheta_2(0) + \beta^2\vartheta_3(0) + \beta^3\vartheta_4(0)}\right] \quad (7c)$$

The values of $\vartheta_i(0)$ and $\vartheta_i'(0)$ that are required in the numerical evaluation of equations (7) are listed in Table 2. Also listed in the table are $f_i''(0)$ values for the velocity solution.

Table 2. Values of $\vartheta(0)$ and $\vartheta'(0)$, Cases (b), (d), (e); Pr = 0·7

i	$-\vartheta_i'(0)$ Case (b)	$\vartheta_i(0)$ Case (d)	Case (e)	$f_i''(0)$
0	0·5854	0	0	1·3282
1	−0·4488	0·8623	0·8623	−1·2243
2	0·04657	0·4621	−0·1662	0·1948
3	0·005822	0·2234	0·00533	0·02822
4		0·1094	−0·000958	

As is usual in series solutions of boundary layer problems, rigorous statements cannot be made about the region of convergence of the series. For the Nusselt number expressions of equations (7), the quantitative contributions of the successive terms of the series were carefully considered. On the basis of such an appraisal, the authors felt that it was reasonable to evaluate Nusselt number results for Cases (b) and (e) for β values as large as unity; on the other hand, for Case (d), the evaluation was terminated at $\beta = 0·7$.

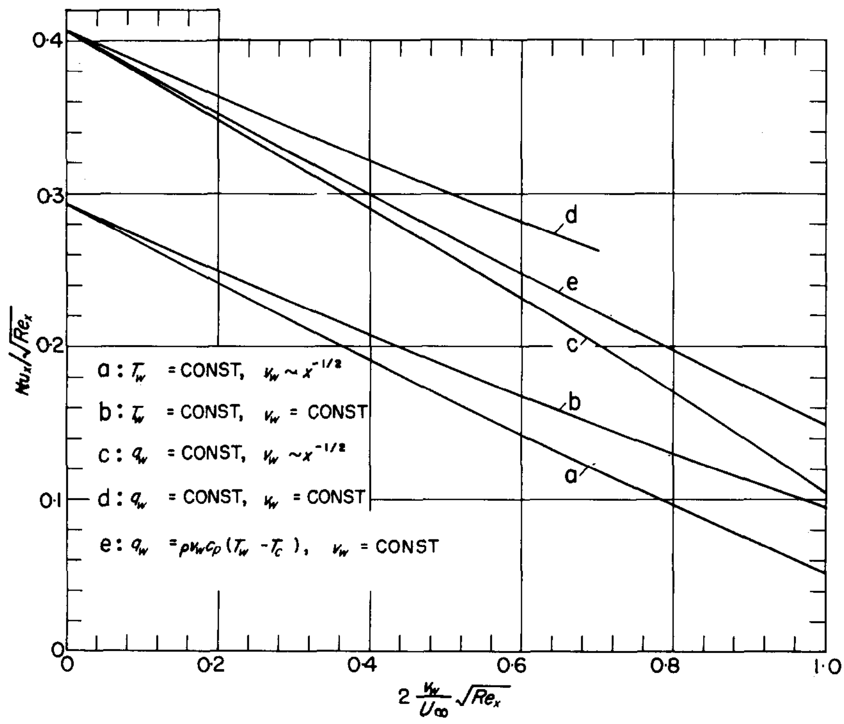

FIG. 1. Nusselt number results.

The Nusselt number results thus obtained are plotted in Fig. 1 as a function of the blowing parameter $\beta = 2(v_w/U_\infty)\sqrt{(Re_x)}$. It is seen that in all cases, the Nusselt number decreases with increasing blowing rate. For Cases (a) and (b), both of which are characterized by T_w = const., the numerical values of $Nu_x/\sqrt{(Re_x)}$ are of comparable magnitude for small and intermediate values of the blowing parameter; however, the deviations grow increasingly larger as the blowing parameter increases. A similar remark applies to the $Nu_x/\sqrt{(Re_x)}$ results for Cases (b), (d), and (e). It is evident that the heat transfer results are quite sensitive to the nature of the thermal and velocity boundary conditions at the plate surface.

abscissa with increasing downstream distances. From the figure, it is seen that as x approaches zero,* T_w approaches T_∞; that is, the transpiration cooling provides no protection of the surface. With increasing values of x, the wall temperature departs more and more from T_∞, tending to approach more and more closely to T_c. Thus, the surface protection provided by transpiration cooling improves with increasing downstream distances. This finding suggests that, for practical application, it is necessary to design the leading edge so that boundary conditions other than those of Case (e) apply in that region. In particular, since v_w is necessarily finite, a reasonable approach would be to provide good paths for

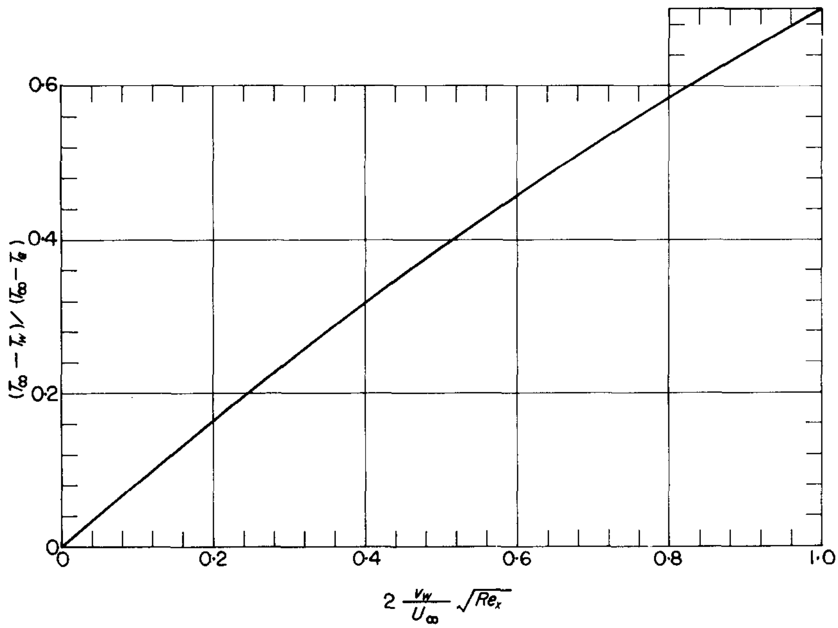

FIG. 2. Variation of plate surface temperature, Case (e).

More detailed consideration will now be given to Case (e) which, as previously noted, is believed to correspond most closely to boundary conditions that are experimentally realistic. Inasmuch as the aim of transpiration cooling is the thermal protection of the surface, it is of interest to inquire about the surface temperature results. This information has been determined by evaluating equation (5) with the aid of Table 2. The results thus obtained are presented in Fig. 2. The dimensionless surface temperature is plotted on the ordinate, while the abscissa is the blowing parameter.

For purposes of discussion, it is convenient to imagine that v_w is fixed and to associate increasing values of the

heat conduction in the wall. In such an event, heat conduction terms would have to be appended to the energy balance that was employed for Case (e). This would necessitate simultaneous solution of the boundary-layer problem and the heat-conduction problem for the wall.

REFERENCE

1. J. P. HARTNETT and E. R. G. ECKERT, *Trans. Am. Soc. Mech. Engrs* **79**, 247 (1957).

* At $x = 0$, the boundary-layer assumptions no longer hold.